Calculating Sums

① Find $\sum_{k=0}^{7} 41+3k$

$$= \underset{k=0}{41} + \underset{k=1}{44} + \underset{k=2}{47} + \underset{k=3}{50} + \underset{k=4}{53} + \underset{k=5}{56} + \underset{k=6}{59} + \underset{k=7}{62}$$

$$= 412.$$

$$= 41\cdot 8 + \underbrace{3(\underbrace{0+1+2+3+4+5+6+7}_{28})}_{84}$$

$$= 328 + 84 = 412.$$

This is an example of an arithmetic progression (but summed up).

$$41 \underset{\text{add } 3}{\curvearrowright} 44 \underset{\text{add } 3}{\curvearrowright} 47 \underset{\text{add } 3}{\curvearrowright} 50 + \cdots \underset{\text{add } 3}{\curvearrowright} 62$$

$$41 + 44 + 47 + 50 + \cdots + 62$$

In summation notation, an arithmetic progression looks like this.

$$\sum_{n=(\#_1)}^{(\#_2)} (\#_3) + n(\#_4)$$

$$\left(\begin{array}{c}\text{Arithmetic Progression}\\ \text{Sum}\end{array}\right) = \left(\overbrace{\frac{(\text{smallest \#}) + (\text{largest \#})}{2}}^{\text{average \#}}\right) \cdot (\text{\# of terms})$$

eg:

$$\sum_{m=4}^{1000} 2m+3 = \underbrace{\left(\frac{11+2003}{2}\right)}_{1007} \cdot (997)$$

$$= \boxed{1{,}003{,}979}$$

Other properties of sums:

- $$\sum_{n=a}^{b} \big(f(n) + g(n)\big) = \sum_{n=a}^{b} f(n) + \sum_{n=a}^{b} g(n)$$

  (commutative & associative properties in action).

- $$\sum_{n=a}^{b} \overset{\text{constant}}{\overset{\downarrow}{c}}\, f(n) = c \sum_{n=a}^{b} f(n)$$

  (distributive property in action).

linearity

eg: $\displaystyle\sum_{m=4}^{1000} 2m+3 = \sum_{m=4}^{1000} 2m + \sum_{m=4}^{1000} 3$

$$= 2\left(\sum_{m=4}^{1000} m\right) + 3\cdot \sum_{m=4}^{1000} 1$$

arithmetic

$\left(\frac{1000+4}{2}\right)\cdot 997$ $\qquad \underbrace{1+1+1+1\ldots 1}_{997}$

$$= 2(502)(997) + 3(997)$$

$$= (1007)(997) = \boxed{1,003,979}$$

Another sum

$$\sum_{k=1}^{N} k^2 = \underset{k=1}{1} + \underset{k=2}{4} + 9 + \dots + N^2$$

$\frac{N(N+1)(2N+1)}{6}$

$1 = 1$ $\qquad \frac{1\cdot 2\cdot 3}{6} = 1$

1  $\qquad 1+4=5$ $\qquad \frac{2(3)(5)}{6} = 5$

4 

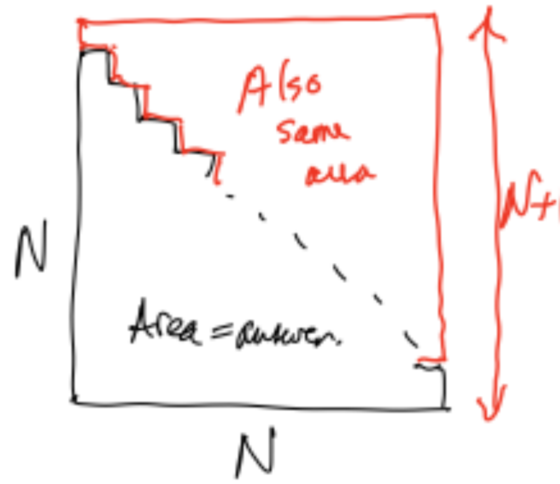
9  $\qquad 1+4+9=14$ $\qquad \frac{3(4)(7)}{6} = 14$

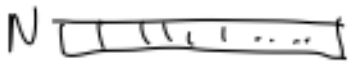
16  $\qquad 1+4+9+16=30$

what is the formula?

- $$\sum_{k=1}^{N} 1 = N$$

- $$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$





- $$\boxed{\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}}$$

Can prove this using "induction"

- $$\boxed{\sum_{k=1}^{N} k^3 = \frac{N^2(N+1)^2}{4}}$$

$N=1$	1	$\frac{1^2\cdot 2^2}{4} = 1$ ✓
$N=2$	$1+8=9$	$\frac{2^2\cdot 3^2}{4} = 9$ ✓
$N=3$	$1+8+27=36$	$\frac{3^2\cdot 4^2}{4} = 36$ ✓
$N=4$	$1+8+27+64=100$	$\frac{4^2\cdot 5^2}{4} = 100$ ✓

$\vdots$

Examples:

(1) $\displaystyle\sum_{k=1}^{20}(2k^2-k^3) = ?$

(2) $\displaystyle\sum_{s=1}^{N}(3s^2+4s-2) = ?$

$$\sum_{k=1}^{N} 1 = N$$

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{k=1}^{N} k^3 = \frac{N^2(N+1)^2}{4}$$

Solutions

(1)
$$\sum_{k=1}^{20}(2k^2-k^3) = \sum_{k=1}^{20} 2k^2 - \sum_{k=1}^{20} k^3$$
$$= 2\left(\sum_{k=1}^{20} k^2\right) - \left(\sum_{k=1}^{20} k^3\right)$$
$$= 2\,\frac{(20)(21)(41)}{6} - \frac{20^2(21)^2}{4} = \boxed{-38{,}360}$$

(2)
$$\sum_{s=1}^{N}(3s^2+4s-2)$$
$$= 3\sum_{s=1}^{N} s^2 + 4\sum_{s=1}^{N} s - 2\sum_{s=1}^{N} 1$$
$$= \boxed{3\,\frac{N(N+1)(2N+1)}{6} + 4\,\frac{N(N+1)}{2} - 2N}$$

Another fun sum to calculate.

Geometric Series:

Eg: Put \$400 into an account each month that earns 2% annual interest, compounded monthly. How much will you have after 27 years?

A geometric series is a sum of the form.

$$S_N = a + ar + ar^2 + ar^3 + \dots + ar^N$$

r = common ratio

a = first term.

$$rS_N = ar + ar^2 + ar^3 + \dots + ar^N + ar^{N+1}$$

Subtract.

$$S_N - rS_N = a - ar^{N+1}$$

$$(1-r)S_N = a - ar^{N+1} = a(1-r^{N+1})$$

$$\Rightarrow \boxed{S_N = \frac{a(1-r^{N+1})}{1-r}}$$ Geometric Series Sum formula.

Back to our question:

Eg: Put \$400 into an account each month that earns 2% annual interest, compounded monthly. How much will you have after 27 years?

Solution:

400 end of 1st month

end of 2nd month: new deposit

$$400 + 400 + (400)\left(\frac{.02}{12}\right) = \underbrace{400}_{\text{new}} + \underbrace{400\left(1+\frac{.02}{12}\right)}_{\text{old + interest}}$$

end of 3rd month

$$\underbrace{400}_{\text{new}} + (\text{old})\cdot\left(1+\frac{.02}{12}\right)$$

$$400 + \left(400 + 400\left(1+\frac{.02}{12}\right)\right)\left(1+\frac{.02}{12}\right)$$

$$= 400 + 400\left(1+\frac{.02}{12}\right) + 400\left(1+\frac{.02}{12}\right)^2$$

end of 4th month

$$= 400 + \underbrace{400\left(1+\frac{.02}{12}\right)}_{\text{since 3rd month + interest}} + \underbrace{400\left(1+\frac{.02}{12}\right)^2}_{\text{since 2nd month + interest}} + \underbrace{400\left(1+\frac{.02}{12}\right)^3}_{\text{since 1st month + interest}}$$

⋮

end of $27\cdot 12^{th}$ month (324^{th} month)

$$= 400 + 400\left(1+\frac{.02}{12}\right) + 400\left(1+\frac{.02}{12}\right)^2 + \cdots + 400\left(1+\frac{.02}{12}\right)^{323}$$

$$= a + ar + ar^2 + \cdots + ar^N$$

$$a = 400,\ r = 1+\frac{.02}{12},\ N = 323$$

Geometric Series

$$\text{Sum} = \frac{a\left(1-r^{N+1}\right)}{1-r} = \frac{400\left(1-\left(1+\frac{.02}{12}\right)^{324}\right)}{1-\left(1+\frac{.02}{12}\right)}$$

$= \$171,656.57$ (approx)

How much was interest?

$\$171,656.57 - 400\cdot 27\cdot 12$

$= \$42,056.57.$